Math 3235 Probability Theory 2/28/23

X is a continuous r.v.

$f(x)$ density (p.d.f.)

$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx$$

$f(x)$ is continuous but for a finite number of points

$$F(x) = \mathbb{P}(X \le x)$$

if X is continuous

$$F(x) = \int_{-\infty}^{x} f(y)\,dy$$

$f(x) \ge 0$ $\qquad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$F(x)$ is not decreasing

$F(-\infty) = 0$ $\qquad F(+\infty) = 1$

Examples

Uniform in $[A, B]$

$$f(x) = \frac{1}{B-A} \qquad A \le x \le B$$

Exponential

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

Standard Normal

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Normal

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Cauchy

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$f(x)$ p.d.f. of X

$$\mathbb{E}(X) = \sum_{x} x\, p(x)$$

Example:

Uniform r.v in $[A, B]$

X

$$f_X(x) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_A^B \frac{x}{B-A}\, dx =$$

$$= \frac{1}{B-A} \left. \frac{x^2}{2} \right|_A^B = \frac{B^2 - A^2}{2(B-A)} =$$

$$= \frac{A+B}{2}$$

$$\mathbb{E}(X) = \frac{A+B}{2}$$ X uniform in $[A, B]$

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so X is exponential par λ

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$\mathbb{E}(X) = \int_0^\infty \lambda x e^{-\lambda x} dx \qquad y = \lambda x$$

$$= \frac{1}{\lambda} \int_0^\infty y e^{-y} dy$$

$$\frac{d}{dy} e^{-y} = - e^{-y}$$

$$\int_0^\infty y e^{-y} dy = \underbrace{-y e^{-y} \Big|_0^\infty}_{=0} + \int_0^\infty e^{-y} dy$$

$$= \int_0^\infty e^{-y} dy = -e^{-y} \Big|_0^\infty = 1$$

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

$$\mathbb{E}(X) = \theta$$

$$\int_0^{\infty} \lambda^n x^{n-1} e^{-\lambda x} dx = \qquad y = \lambda x$$

$$= \int_0^{\infty} y^{n-1} e^{-y} dy = \Gamma(n)$$

$$\Gamma(n+1) = \int_0^{\infty} y^n e^{-y} dy =$$

$$= -\underbrace{y^n e^{-y} \Big|_0^{\infty}}_{0} + n \int_0^{\infty} y^{n-1} e^{-y} dy$$

$$= n\Gamma(n)$$

$$\Gamma(n+1) = n\Gamma(n)$$

If n is integer

$$\Gamma(n) = (n-1)!$$

$\Gamma(\alpha)$ is defined for every $\alpha > 0$.

$\Gamma(\alpha)$ is a way to extend the factorial to non-integers.

$0 \longrightarrow x$

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}$$

$$\int_0^{\infty} f(x)\,dx = 1 \qquad f(x) > 0$$

$f(x)$ is called a $\Gamma(\alpha, \beta)$ p. d. f.

$\Gamma(1, \beta)$ is exponential of par β.

Standard Normal

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x e^{-\frac{x^2}{2}}}_{g(x)} dx = 0$$

$$g(-x) = -g(x)$$

$$0$$

Cauchy

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$$

$$\frac{1}{\pi} \int_{-M}^{L} \frac{x}{1+x^2} dx \qquad L, M \to \infty$$

$$= \frac{1}{2\pi} \left(\ln(1+L^2) - \ln(1+M^2) \right)$$

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx \quad \text{is not defined!!}$$

If X is Cauchy, $\mathbb{E}(X)$ does not exist.

$$\frac{1}{\sqrt{2\pi}} \int_{-L}^{M} x e^{-\frac{x^2}{2}}\,dx = -\frac{1}{2}\frac{1}{\sqrt{2\pi}} \left(e^{-\frac{M^2}{2}} - e^{-\frac{L^2}{2}} \right)$$

$f(x)$ $\qquad N(\mu, \sigma^2)$

$$\mathbb{E}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$x - \mu = y$

$$\mathbb{E}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int (y+\mu) e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int y \, e^{-\frac{y^2}{2\sigma^2}} \, dy \qquad = 0$$

$$+ \frac{\mu}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} \, dy$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} \, dy = \qquad z = y/\sigma$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz = 1$$

if X is $\mathcal{N}(\mu, \sigma^2)$ Than

$$\mathbb{E}(X) = \mu$$

$$z = \frac{x - \mu}{\sigma}$$

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if h is a function $\mathbb{R} \to \mathbb{R}$

$Y = h(X)$ is a r.v.

$$\mathbb{E}(Y) = \mathbb{E}(h(X)) = \int_{-\infty}^{\infty} h(x) f(x) \, dx$$

$$\mathbb{E}(Y) = \sum_x h(x)\, p(x)$$

$$Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) =$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$Var(aX + b) = a^2\, Var(X)$$

$$\mathbb{E}(aX + b) = a\,\mathbb{E}(X) + b$$

for $a, b \in \mathbb{R}$.

$Var(X) = \sigma_X^2$ $\qquad$ σ_X standard deviation.

X is uniform in $[A, B]$

$$\mathbb{E}(X^2) = \int_A^B \frac{x^2}{B - A}\, dx = \frac{x^3}{3} \frac{1}{B - A} =$$

$$= \frac{B^3 - A^3}{3\, B - A}$$

$$Var(X) = \frac{B^3 - A^3}{3B - A} - \frac{(B+A)^2}{4} =$$

$$= \frac{1}{12} \frac{4(B^3 - A^3) - 3(B-A)(B+A)^2}{B - A} =$$

$$= \frac{(B-A)^2}{12}$$

$$Var(X) = \frac{(B-A)^2}{12}$$

if X is uniform in $[0,1]$

$\Downarrow$

$Y = A + (B-A)X$ is uniform in $[A,B]$

$$Var(Y) = (B-A)^2 \, Var(X)$$